JOURNAL OF NUMBER THEORY 35, 109–127 (1990)

Algebraic Groups and Small Transcendence Degree, II

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Communicated by M. Waldschmidt

Received December 31, 1987; revised June 20, 1989

We continue our investigation into the algebraic independence of two numbers associated with a one-parameter subgroup of a commutative algebraic group. Here we focus on numbers which are associated with either algebraic points or torsion points on the algebraic group. © 1990 Academic Press, Inc.

In [11] we investigated the algebraic independence of two numbers associated with a one-parameter subgroup of a commutative algebraic group defined over an arbitrary subfield of the complex numbers. The results of [11] were rather general (see below); no hypotheses were made concerning the arithmetic nature of the values under consideration. In this paper we investigate the algebraic independence of numbers associated with algebraic and/or torsion points of a commutative algebraic group. We retain several features of [11], in particular our numbers are associated with a one-parameter subgroup of an algebraic group, and the algebraic group is given as the product of a general commutative algebraic group and powers of the additive and the multiplicative groups of complex numbers.

Specifically, suppose that G is a commutative algebraic group of dimension $d \geqslant 1$ which is defined over an arbitrary subfield K of $\mathbf{C}$. Suppose further that

$$G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1} \times G_2, \tag{1}$$

where $\mathbf{G}_a^{d_0}$ (respectively, $\mathbf{G}_m^{d_1}$) is the maximal unipotent (respectively, multiplicative) factor of G, and G_2 is defined over K with $\dim G_2 = d_2$. Note that $d = d_0 + d_1 + d_2$.

Let $\phi: \mathbf{C} \to G(\mathbf{C})$ be an analytic homomorphism with $\phi(\mathbf{C})$ Zariski dense in $G(\mathbf{C})$; and, let $\mathscr{T}_G(\mathbf{C})$ denote the tangent space of $G(\mathbf{C})$ at its identity

* Research supported in part by a grant from the National Science Foundation.

0022-314X/90 $3.00

element. Then there exists a linear map $L: \mathbf{C} \to \mathcal{T}_G(\mathbf{C})$ such that $\phi(z) = \exp_G \circ L(z)$; here $\exp_G$ is the exponential map of G, $\exp_G$: $\mathcal{T}_G(\mathbf{C}) \to G(\mathbf{C})$. Denote the dimension of the vector space of least dimension which is defined over K and which contains $L(\mathbf{C})$ by n.

Assume that $y_1, ..., y_l$ are $\mathbb{Z}$-linearly independent complex numbers such that

$$Y = y_1\mathbb{Z} + \cdots + y_l\mathbb{Z} \tag{2}$$

satisfies $\phi(Y) \subseteq G(K)$. The general problem in [11] was to find conditions on l, n, d_0, d_1, and d_2 which imply that K contains at least two algebraically independent numbers. It was shown there that if

$$l(d-n) \geqslant l + 2d_2 + d_1 + \varepsilon \tag{3}$$

with

$$\varepsilon = \begin{cases} 1 & \text{if} \quad d_0 = n = 1 \\ 0 & \text{otherwise} \end{cases}$$

then

$$\text{trans deg}_{\mathbf{Q}} K \geqslant 2.$$

Moreover, if $Y \cap \ker \phi \neq 0$ and

$$2l(d-n) \geqslant l + 2d_2 + d_1 \tag{4}$$

then

$$\text{trans deg}_{\mathbf{Q}}\, K \geqslant 2.$$

One goal of this paper is to study the algebraic independence of values associated with algebraic points on a commutative algebraic group defined over $\bar{\mathbf{Q}}$. We will see below that in several cases relating to algebraic points the inequality (3) above with $\varepsilon = 0$ implies that $\text{trans deg}_{\mathbf{Q}} K \geqslant 2$ even though $d_0 = n = 1$. These results have several consequences concerning numbers associated with algebraic numbers or algebraic points on an elliptic curve; and, among other things provide generalizations of the so-called Brownawell–Waldschmidt Theorem (see [1, 15], or Section 2 below).

To give these results a uniform formulation suppose that we have a more refined decomposition of G than in (1). More precisely, suppose that

$$G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1} \times G_2' \times G_2, \tag{5}$$

where G_2' is a commutative algebraic group defined over K with $\dim G_2' = d_g$, and

$$G_2 = \mathbf{G}_a^{\delta_a} \times \mathbf{G}_m^{\delta_m} \times G_2'', \tag{6}$$

where G_2'' is defined over $\bar{\mathbf{Q}}$ and $\dim G_2''=\delta_g$. In particular, we no longer assume that $\mathbf{G}_a^{d_0}$ (resp., $\mathbf{G}_m^{d_1}$) is the maximal unipotent (resp., multiplicative) factor of G but that $\mathbf{G}_a^{d_0}\times\mathbf{G}_a^{\delta_a}$ (resp. $\mathbf{G}_m^{d_1}\times\mathbf{G}_m^{\delta_m}$) is. Note that with $d_2=d_g+\delta_a+\delta_m+\delta_g$ we still have $d=d_0+d_1+d_2$.

Suppose that $\phi:\mathbf{C}\to G(\mathbf{C})$ is an analytic homomorphism with $\phi(\mathbf{C})$ Zariski dense in $G(\mathbf{C})$. Suppose moreover that $\phi'(0)\in\mathscr{T}_G(K)$; i.e., in the notation above $n=1$. With Y as in (2) above we also assume that $\phi(Y)\subseteq G(K)$. We note that since $\phi(\mathbf{C})$ is Zariski dense in $G(\mathbf{C})$ it follows that $d_0+\delta_a\leqslant 1$.

We begin with a result which concerns the situation where all of the points under consideration are associated with algebraic points on G_2. Throughout this paper $\pi: G\to G_2$ will be the canonical projection mapping.

THEOREM 1. *If* $\dim G_2>\delta_a$ *and*

$$\pi\circ\phi(y_i)\in G_2(\bar{\mathbf{Q}})\qquad(1\leqslant i\leqslant l)$$

then

$$l(d-1)\geqslant l+2(d_g+\delta_g)+(d_1+\delta_m)\tag{7}$$

implies that

$$\text{trans deg}_{\mathbf{Q}}\,K\geqslant 2.$$

The next theorem involves only one algebraic point, but requires additional hypotheses on G.

THEOREM 2. *Suppose that* $d_g=0$ *and* $d>d_1+2$. *If*

$$\pi\circ\phi(y_1)\in G_2(\bar{\mathbf{Q}})$$

then

$$l(d-1)\geqslant l+2\delta_g+(d_1+\delta_m)\tag{8}$$

implies that

$$\text{trans deg}_{\mathbf{Q}}\,K\geqslant 2.$$

These theorems can be seen to be refinements of the results of [11] in that inequality (7) or (8) reduces to (3) with $\varepsilon=0$ when we take $d_g=0$ and $G_2=G_2''$.

In a slightly different direction we also consider values associated with torsion points on G_2 in the decomposition of G given by (5) and drop the hypothesis that G_2 is defined over $\bar{\mathbf{Q}}$. If $y_1,\ldots,y_h$ are $\mathbb{Z}$-linearly independent complex numbers with

$$\pi\circ\phi(y_i)\in G_2(K)_{tors},\qquad 1\leqslant i\leqslant h,$$

then there exist nonzero integers $n_1, \ldots, n_h$ such that

$$n_i y_i \in \ker(\pi \circ \phi), \qquad 1 \leqslant i \leqslant h.$$

Since $\pi \circ \phi: \mathbf{C} \to G_2(\mathbf{C})$ is an analytic homomorphism, whose image is not trivial, it follows that $h \leqslant 2$. We take the cases $h=1$ or $h=2$ into account in our next theorems.

THEOREM 3. *Assume that G is defined over K with $d_1 = d_g = 0$ and that ϕ and Y are as before with $d_0 = n = 1$. Suppose further that $2\delta_m + \delta_g \geqslant 3$. If $\phi(Y) \subseteq G(K)$ and*

$$\pi \circ \phi(y_1) \in G_2(K)_{tors},$$

then $l \geqslant 2$ implies that

$$\text{trans deg}_{\mathbf{Q}}\, K \geqslant 2.$$

When $h=2$ in the above notation, $\pi \circ \phi(z)$ is a doubly periodic function of a single complex variable. Consequently we may as well assume that $G_2 = E$ is an elliptic curve.

THEOREM 4. *Assume that $G = \mathbf{G}_a \times \mathbf{G}_m \times E$ with $n=1$ and E defined over K, and that ϕ is as above. Suppose that y_1, y_2 are $\mathbb{Z}$-linearly independent with*

$$\pi \circ \phi(y_i) \in E(K)_{tors}, \qquad i = 1, 2.$$

If

$$\phi(y_i) \in G(K), \qquad i = 1, 2,$$

then

$$\text{trans deg}_{\mathbf{Q}}\, K \geqslant 2.$$

To conclude this introduction we thank Michel Waldschmidt for his many helpful comments on this project and the referee of an earlier version of this paper.

1. APPLICATIONS

Let $\wp(z)$ be a Weierstrass elliptic function with lattice of periods $\mathscr{L} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and with invariants g_2, g_3. Let $\sigma(z)$ denote the associated Weierstrass sigma function and let

$$h(z) = \sigma^3(z), \qquad f(z) = \sigma^3(z)\, \wp(z), \qquad g(z) = \sigma^3(z)\, \wp'(z).$$

Then $\mathbf{p}(z) = (h(z), f(z), g(z))$ gives a parametrization of an elliptic curve E defined over $\mathbf{Q}(g_2, g_3)$.

COROLLARY 1. *Suppose that u and v are $\mathbb{Z}$-linearly independent complex numbers with $\mathbf{p}(u) \in E(K)_{tors}$. Then at least two of the finite values among*

$$g_2, g_3, \pi, u, v, e^{(\pi i/u)v}, \wp(v)$$

are algebraically independent.

Proof. Let $G = \mathbf{G}_a \times \mathbf{G}_m \times E$ and define $\phi: \mathbf{C} \to G(\mathbf{C})$ by

$$\phi(z) = (1, z, 1, e^{(\pi i/u)z}, \mathbf{p}(z)).$$

The algebraic independence of the functions z, $e^{(\pi i/u)z}$, and $\wp(z)$ over $\mathbf{C}$ implies that $\phi(\mathbf{C})$ is Zariski dense in $G(\mathbf{C})$. Thus the hypotheses of Theorem 3 are satisfied with $y_1 = u$ and $y_2 = v$.

For example, suppose that g_2 and g_3 are algebraic. If α is an algebraic number, $\alpha \neq 0, 1$, such that $(\log \alpha)/\pi i \notin \mathbf{Q}$, put $v = (u/\pi i) \log \alpha$ in Corollary 1. Then at least two of

$$u, \pi, \log \alpha, \wp\left(\frac{u}{\pi i} \log \alpha\right)$$

are algebraically independent.

In the next three applications we retain the hypothesis that g_2 and g_3 are algebraic.

COROLLARY 2. *Suppose that u_1, u_2, u_3 are $\mathbb{Z}$-linearly independent complex numbers with $\mathbf{p}(u_i) \in E(\bar{\mathbf{Q}})$ for $i = 1, 2, 3$. Then for any $\beta \neq 0$ at least two of*

$$\beta, u_1, u_2, u_3, e^{\beta u_1}, e^{\beta u_2}, e^{\beta u_3}$$

are algebraically independent.

Proof. Take G and ϕ as in the proof of Corollary 1, with $e^{\beta z}$ replacing $e^{(\pi i/u)z}$. The hypotheses of Theorem 1 are satisfied and the corollary follows.

For $\wp(z)$ as above let $\tau = \omega_2/\omega_1$; $\wp(z)$ has complex multiplications precisely when τ is imaginary quadratic. Put $K_\tau = \mathbf{Q}(\tau)$ in this case and $K_\tau = \mathbf{Q}$ otherwise. The significance of K_τ lies in the fact that $\wp(z)$ and $\wp(\alpha z)$ are algebraically dependent functions over $\mathbf{C}(z)$ exactly when $\alpha \in K_\tau$.

COROLLARY 3. *Suppose that $\wp(z)$ has complex multiplication and that g_2 and g_3 are algebraic. Then at least two of*

$$\omega_1, e, e^{\tau}$$

are algebraically independent.

Proof. Apply Theorem 4 to $G = \mathbf{G}_a \times \mathbf{G}_m \times E$ with $y_1 = 1$, $y_2 = \tau$, and

$$\phi(z) = (1, z, 1, e^z, \mathbf{p}(\omega_1 z)).$$

Note that it more generally follows from Theorem 4 that if $\wp(z)$ is a Weierstrass $\wp$-function, then for any $\lambda \neq 0$ at least two of

$$g_2, g_3, \omega_1, \omega_2, \lambda, e^{\lambda\omega_1}, e^{\lambda\omega_2}$$

are algebraically independent.

COROLLARY 4. *Suppose that u_1 and u_2 are $\mathbb{Z}$-linearly independent complex numbers with*

$$\mathbf{p}(u_i) \in E(\bar{\mathbf{Q}}) \qquad \textit{for} \quad i = 1, 2.$$

Then for any $\mathbb{Z}$-linearly independent complex numbers β_1, β_2 at least two of the finite values among

$$\beta_1, \beta_2, u_1, u_2, e^{\beta_1 u_1}, e^{\beta_1 u_2}, e^{\beta_2 u_1}, e^{\beta_2 u_2}$$

are algebraically independent.

Proof. This is an application of Theorem 1, where we take $G = \mathbf{G}_a \times \mathbf{G}_m \times \mathbf{G}_m \times E$ and

$$\phi(z) = (1, z, 1, e^{\beta_1 z}, e^{\beta_2 z}, \mathbf{p}(z)).$$

One consequence of Corollary 4 is that if $\wp(z)$ has complex multiplication by $\rho \in K_\tau \backslash \mathbf{Q}$ with $\rho^2 \in \mathbf{Q}$, and $\wp(u)$ is algebraic, then at least two of

$$u, e^u, e^{\rho u}$$

are algebraically independent (put $u_1 = u$, $u_2 = \rho u$, $\beta_1 = 1$, and $\beta_2 = \rho$).

Remarks. (1) Theorem 4 generalizes two results which have appeared in the literature: [3, Theorem 4.1(i)] and [10, Theorem].

(2) Corollary 4 *without* the hypotheses $g_2, g_3 \in \bar{\mathbf{Q}}$ has been claimed by Chudnovsky [3, Theorem 4.8, p. 316] but the proof he indicates is insufficient. The conclusion there is that two of $g_2, g_3, \beta_j, u_i, e^{\beta_j u_i}$, $j = 1, 2$ and $i = 1, 2$, are algebraically independent.

(3) Theorem 1 implies part (b) of the main theorem of [12], and hence Corollaries 1 and 2 of that paper.

(4) The main results of [11] (inequalities (3) and (4) above) could be applied in each corollary of this section, but in every case the conclusion would be slightly weaker (i.e., more numbers would be required to generate the field K).

2. The Brownawell–Waldschmidt Theorem

In 1971 Brownawell [1] and Waldschmidt [15] independently proved that either e^e or e^{e^2} is transcendental. This had been conjectured by T. Schneider [8]. That one of their values is transcendental is included in the following result which they proved.

Corollary 5 (Brownawell, Waldschmidt). *Let u_1, u_2 and v_1, v_2 be $\mathbb{Z}$-linearly independent complex numbers with $e^{v_1 u_1}$ and $e^{v_2 u_1}$ algebraic. Then at least two of*

$$u_1, u_2, v_1, v_2, e^{v_1 u_2}, e^{v_2 u_2}$$

are algebraically independent.

Proof. (Using Theorem 1) Let $G = \mathbf{G}_a \times \mathbf{G}_m \times \mathbf{G}_m$ with $\phi(z) = (z, e^{u_2 z}, e^{u_1 z})$, where we view $\mathbf{G}_m$ as G_2. Then by Theorem 1, since $e^{u_1 v_1}$ and $e^{u_1 v_2}$ are algebraic, we need $l \geqslant 2$ points so we take $y_1 = v_1$ and $y_2 = v_2$.

(Using Theorem 2) Let $G = \mathbf{G}_a \times \mathbf{G}_m \times \mathbf{G}_m$ with $\phi(z) = (z, e^{v_1 z}, e^{v_2 z})$ where we view $\mathbf{G}_m \times \mathbf{G}_m$ as G_2. By Theorem 2, since $e^{v_1 u_1}$ and $e^{v_2 u_1}$ are algebraic, we need $l \geqslant 2$ points, so we take $y_1 = u_1$ and $y_2 = u_2$.

Two elliptic analogues to the Brownawell–Waldschmidt theorem have been given. The first of these was established by Masser and Wüstholz in [4] and the second by the present author in [12]. The former of these is a consequence of Theorem 1 and the latter a consequence of Theorem 2. These results share the common hypotheses that $u_1, ..., u_4$ are $\mathbb{Z}$-linearly independent complex numbers and that v_1, v_2 are K_τ-linearly independent complex numbers.

Corollary 6 [4, Theorem 5]. *With u_i, v_j ($1 \leqslant i \leqslant 4$, $1 \leqslant j \leqslant 2$) as above suppose that each of*

$$g_2, g_3, \wp(v_1 u_1), \wp(v_1 u_2), \wp(v_1 u_3), \wp(v_1 u_4)$$

is algebraic. Then at least two of the finite values among

$$u_i,\ v_j,\ \wp(u_i v_j) \qquad (1 \leqslant i \leqslant 4,\ 1 \leqslant j \leqslant 2) \tag{9}$$

are algebraically independent.

Proof. Let E denote the elliptic curve associated with $\wp(z)$, where we view E as G_2, and put $\phi(z) = (z, \mathbf{p}(v_2 z), \mathbf{p}(v_1 z))$. Theorem 1 then implies the corollary.

Slightly later the following was established.

COROLLARY 7 [12, Theorem (b)]. *With u_i, v_j $(1 \leqslant i \leqslant 4,\ 1 \leqslant j \leqslant 2)$ as above suppose that each of*

$$g_2,\ g_3,\ \wp(v_1 u_1),\ \wp(v_2 u_1)$$

is algebraic. Then at least two of the finite values in (9) *are algebraically independent.*

Proof. This is an application of Theorem 2 where we take $G = \mathbf{G}_a \times E \times E$ $(G_2 = E \times E)$ with $\phi(z) = (z, \mathbf{p}(v_1 z), \mathbf{p}(v_2 z))$.

The hypotheses of Corollaries 6 and 7 are clearly different, in Corollary 6 you assume that one function is algebraic at several points and in Corollary 7 you assume that several functions are algebraic at one point; yet both provide elliptic analogues to Corollary 5. It follows that both Theorem 1 and Theorem 2 may be thought of as generalizations of the Brownawell–Waldschmidt theorem.

3. THE PROOF OF THEOREM 1

Assume that the theorem is false. If $K \subseteq \bar{\mathbf{Q}}$, then G is defined over $\bar{\mathbf{Q}}$ and the condition $\phi(Y) \subseteq G(K)$ together with (7) contradicts Théorème 3.1.1 of [13]. Thus K may be expressed as $K = \mathbf{Q}(\theta, \eta)$ where θ is transcendental and η is integral over $\mathbb{Z}[\theta]$. Put $\mathcal{O}_K = \mathbb{Z}[\theta, \eta]$. For $\alpha = P(\theta, \eta) \in \mathcal{O}_K$ with $P(x, y) \in \mathbb{Z}[x, y]$ let $\deg \alpha = \deg_x P$ and $ht\alpha = htP$, where htP denotes the maximum absolute value of the coefficients of P.

Let $G_2' \to^f \mathbf{P}_N$ and $G_2 \to^{\hat{f}} \mathbf{P}_{\hat{N}}$ be the embeddings given by J-P. Serre [9]. Then G has a natural embedding into the multiprojective space

$$\mathbf{P}_{d_0 + \delta_a} \times \mathbf{P}_{d_1} \times \mathbf{P}_N \times \mathbf{P}_{\delta_m} \times \mathbf{P}_{\hat{N}}.$$

The coordinate functions $h_0, \ldots, h_N$ of $\mathbf{h} = f \circ \exp_{G_2'}$ and $\hat{h}_0, \ldots, \hat{h}_{\hat{N}}$ of $\hat{\mathbf{h}} = \hat{f} \circ \exp_{G_2''}$ are entire functions with orders of growth at most two. Moreover, since $\phi'(0) \in \mathcal{T}_G(K)$ the ring

$$K\left[\frac{h_0}{h_j}(L(z)), ..., \frac{h_N}{h_j}(L(z))\right], \qquad \left(\text{resp., } K\left[\frac{\hat{h}}{\hat{h}_j}(L(z)), ..., \frac{\hat{h}_{\hat{N}}}{\hat{h}_j}(L(z))\right]\right)$$

is stable under d/dz whenever $h_j \neq 0$ (resp., $\hat{h}_j \neq 0$). (For details see, e.g., [5].)

Since $n=1$, there exist $\alpha, \beta_1, ..., \beta_{d_1}, \gamma_1, ..., \gamma_{\delta_m}$ in K, and without loss of generality in $\mathcal{O}_K$, such that when G is viewed as embedded into the multiprojective space above $\phi(z)$ may be represented in the form

$$\begin{aligned}\phi(z) = (1, \alpha z, 1, \exp(\beta_1 z), ..., \exp(\beta_{d_1} z), \mathbf{h}(L(z)),\\ 1, \exp(\gamma_1 z), ..., \exp(\gamma_{\delta_m} z), \hat{\mathbf{h}}(L(z))).\end{aligned} \tag{10}$$

LEMMA 3.1.[1] *Suppose that G is a commutative algebraic group, $G \to^f \mathbf{P}_N$ is the embedding given by Serre [9], and $\mathbf{h} = (h_0, ..., h_N)$ is the mapping defined by $\mathbf{h} = f \circ \exp_G$. Let $t_1, ..., t_l$ be elements of $\mathcal{T}_G(\mathbf{C})$ and $s_1, ..., s_l$ be nonnegative integers; and suppose that K is a subfield of* $\mathbf{C}$ *such that $\exp_G(t_i) \in G(K)$ for $1 \leqslant i \leqslant l$.*

If G is defined over $\bar{\mathbf{Q}}$ and $\exp_G(t_i) \in G(\bar{\mathbf{Q}})$ for $1 \leqslant i \leqslant r$, then $\mathbf{h}(s_1 t_1 + \cdots + s_l t_l)$ may be given projective coordinates in $\mathcal{O}_K$ with degrees at most $c_1 \max_{r+1 \leqslant i \leqslant l} s_i^2$ and logarithmic heights at most $c_2(1 + \max_{1 \leqslant i \leqslant l} s_i^2)$.

If G is defined over K and $\exp_G(t_i) \in G(K)_{tors}$ for $1 \leqslant i \leqslant r$, then $\mathbf{h}(s_1 t_1 + \cdots + s_l t_l)$ may be given projective coordinates in $\mathcal{O}_K$ with degrees and logarithmic heights at most $c_3(1 + \max_{r+1 \leqslant i \leqslant l} s_i^2)$.

Proof. We apply Proposition 2.2 of [11] with the additional observations that when G is defined over $\bar{\mathbf{Q}}$ and $\exp_G(t) \in G(\bar{\mathbf{Q}})$, then $\mathbf{h}(st)$ may be represented by projective coordinates in $\mathcal{O}_K \cap \bar{\mathbf{Q}}$ with absolute values at most $c_4(1+s^2)$; and if G is defined over K with $\exp_G(t) \in G(K)_{tors}$, then for some positive integer n, $nt \in \ker(\exp_G)$ and therefore the projective coordinate of $\mathbf{h}(st)$ may be taken in $\mathcal{O}_K$ with degree at most c_5 and logarithmic heights at most $c_6(1+s^2)$.

As is standard in transcendence proofs various parameters must be specified. Put

$$\delta = \frac{l + 2(d_g + \delta_g) + (d_1 + \delta_m)}{d-1} \quad \text{and} \quad \rho = -\frac{\delta_m + \delta_g}{d-1}.$$

Then for $S > 0$ define parameters $L_0, L_1, L_2, T, l_1, l_2$ by

$$L_0 = [\![S^\delta \log^\rho S]\!] \qquad L_1 = [\![S^{\delta-1} \log^\rho S]\!]$$

$$L_2 = [\![S^{\delta-2} \log^\rho S]\!] \qquad T = [\![\kappa_1 S^\delta \log^\rho S]\!]$$

[1] The constants $c_1, ..., c_{40}$ which appear in the remainder of this paper are effective and depend at most on G, $y_1, ..., y_l$, and the embedding of G into multiprojective space.

with κ_1 chosen below, and

$$l_1 = [\![S^{\delta-1}\log^{\rho+1} S]\!] \qquad l_2 = [\![S^{\delta-2}\log^{\rho+1} S]\!].$$

Then put

$$\delta(S) = L_0 + L_1 S + L_2 S^2 + l_1 + l_2 + T$$
$$\gamma(S) = L_0 \log S + L_1 S + L_2 S^2 + l_1 S + l_2 S^2 + T \log S.$$

Finally, let

$$Y(S) = y_1 \mathbb{Z}(S) + \cdots + y_l \mathbb{Z}(S)$$

and let $\mathscr{G}$ denote the multihomogeneous ideal which defines G in multiprojective space.

We prove the theorem under the hypotheses that $d_1 \leqslant 1$ and $\delta_m \leqslant 1$ from which the general result follows, since δ is a monotonically decreasing function in each of d_1 and δ_m.

PROPOSITION 3.2. *If the hypotheses of Theorem* 1 *hold and* trans $\deg_{\mathbf{Q}} K = 1$, *then there exists a constant* $C_1 > 0$ *such that for all* $S > C_1$ *the following holds. There exists a multihomogeneous polynomial*

$$\mathbf{P}(\mathbf{Z}_1, \mathbf{Y}_1, \mathbf{X}_1, \mathbf{Z}_2, \mathbf{Y}_2, \mathbf{X}_2)$$

of multidegree $(d_0 L_0, d_1 L_1, d_g L_2, \delta_a L_0, \delta_m l_1, \delta_g l_2)$, *with coefficients in* $\mathcal{O}_K$ *of degree at most* $c_7 \delta(S)$ *and logarithmic heights at most* $c_8 \gamma(S)$, *such that*

$$F(z) = P \circ \phi(z)$$

satisfies

$$F^{(t)}(y) = 0, \qquad 0 \leqslant t < T,\ y \in Y(S). \tag{11}$$

Moreover, P does not vanish identically on G.

Proof. Let V be an indexing set for a set of multihomogeneous monomials of multidegree $(d_0 L_0, d_1 L_1, d_g L_2, \delta_a L_0, \delta_m l_1, \delta_g l_2)$ which are linearly independent modulo $\mathscr{G}$; for $v \in V$, $v = (i_1, j_1, k_1, i_2, j_2, k_2)$ let

$$M_v = \mathbf{Z}_1^{i_1} \mathbf{Y}_1^{j_1} \mathbf{X}_1^{k_1} \mathbf{Z}_2^{i_2} \mathbf{Y}_2^{j_2} \mathbf{X}_2^{k_2}$$

Note that there are at least $c_9 L_0^{d_0+\delta_a} L_2^{d_1} L_2^{d_g} l_1^{\delta_m} l_2^{\delta_g}$ such monomials. Define P by

$$P(\mathbf{Z}_1, \mathbf{Y}_1, \mathbf{X}_1, \mathbf{Z}_2, \mathbf{Y}_2, \mathbf{X}_2) = \sum_{v \in V} a_v M_v$$

with undetermined coefficients a_v. As in the proof of Lemma 4.1 of [11] the strategy is to treat the conditions (11) as a system of homogeneous equations with unknowns a_v and coefficients in $\mathcal{O}_K$. It is essential to have arithmetic estimates for the coefficients of those unknowns and this is where Lemma 3.1 plays a role.

For $y \in Y$ put $G_y(z) = F(z+y)$. It follows, as in the proof of Lemma 4.1 in [11], that there exist multihomogeneous polynomials $P_{v,y}$ of multidegree at most $(c_{10}d_0L_0, c_{10}d_1L_1, c_{10}d_gL_2, c_{10}\delta_aL_0, c_{10}\delta_ml_1, c_{10}\delta_gl_2)$, with coefficients in $\mathcal{O}_K$ $[\alpha y, \exp(\beta_1 y), ..., \exp(\beta_{d_1} y), \mathbf{h}(L(y)), \exp(\gamma_1 y), ..., \exp(\gamma_{\delta_m} y), \hat{\mathbf{h}}(L(y))]$ such that

$$G_y(z) = c_{11} \sum_{v \in V} a_v P_{v,y}(\phi(z)),$$

where $c_{11} \neq 0$. Moreover, by Lemma 3.1 we conclude that $\hat{\mathbf{h}}(L(s_1 y_1 + \cdots + s_l y_l))$ may be given projective coordinates in $\mathcal{O}_K$ of degree (respectively, logarithmic height) at most

$$c_{12} \qquad \text{(respectively, } c_{13}(1 + \max_{1 \leqslant i \leqslant l} s_i^2)).$$

Hence for $y \in Y(S)$ the coefficients of $P_{v,y}$ may be taken in $\mathcal{O}_K$ with

$$\deg_\theta \leqslant c_{14}\delta(S), \qquad \log ht \leqslant c_{15}\gamma(S).$$

Let $\hat{d} = \deg_{X_2} P_{v,y}$ and choose $\hat{h}_i(z)$ such that $\hat{h}_i(0) \neq 0$. Then there exists a nonzero constant c_{16} such that

$$c_{16}\left(\frac{d}{dz}\right)^t \left[\frac{G_y(z)}{\hat{h}_i^{\hat{d}}(z)}\right]_{z=0} = \sum_{v \in V} a_v b_{tvy}$$

with b_{tvy} in $\mathcal{O}_K$ having $\deg b_{tvy} \leqslant c_{17}(\delta(S) + t)$ and $\log ht\, b_{tvy} \leqslant c_{18}(\gamma(S) + t \log S)$. It is seen in Siegel's lemma that there exists a constant $c(K)$ such that the system of equations

$$\sum_{v \in V} a_v b_{tvy} = 0, \qquad 0 \leqslant t < T,\ y \in Y(S)$$

may be solved with a_v satisfying the proposition, provided

$$c_9 L_0^{d_0+\delta_a} L_1^{d_1} L_2^{d_g} l_1^{\delta_m} l_2^{\delta_g} \geqslant c(K)\, TS^l. \tag{12}$$

We choose κ_1 in our definition of T above so that equality holds in (12) and the proposition is established.

Before deducing the central corollary from this proposition we introduce some notation. For a variety $V \subseteq \mathbf{P}_{N_1} \times \cdots \times \mathbf{P}_{N_k}$ and integers $t_1 \geqslant 0, ...,$

$t_k \geqslant 0$ let $H(V; t_1, ..., t_k)$ denote the maximal number of monomials in variables $X_{0,1}, ..., X_{N_1,1}, ..., X_{0,k}, ..., X_{N_k,k}$ of multidegree $(t_1, ..., t_k)$ which are linearly independent modulo the ideal defining V. Moreover, let deg V denote the maximal number of points of intersection of V and dim V hyperplanes. Then when $V = V_1 \times \cdots \times V_k$ for subvarieties V_i of $\mathbf{P}_{N_i}$, Lemme 3.4 of [5] yields the representation

$$H(V; t_1, ..., t_k) = \frac{(\dim V)!}{(\dim V_1)! \cdots (\dim V_k)!} \deg V_1 \cdots \deg V_k t_1^{\dim V_1} \cdots t_k^{\dim V_k}. \tag{13}$$

This quantity plays a crucial role in what follows, as does the notion of multidegree.

For $\mathbf{a} = (a_1, ..., a_k)$ with $0 \leqslant a_1 \leqslant N_1, ..., 0 \leqslant a_k \leqslant N_k$ with $a_1 + \cdots + a_k =$ dim V let

$$\deg_{\mathbf{a}} V = \max\{\operatorname{card}(V \cap L_1 \times \cdots \times L_k)\},$$

where L_i runs over all linear subvarieties of $\mathbf{P}_{N_i}$ of codimension a_i such that $\dim(V \cap L_1 \times \cdots \times L_k) = 0$. Put $\deg_{\mathbf{a}} V = 0$ if no such subvarieties exist.

COROLLARY 3.3. *Under the hypotheses of Theorem* 1, *if* trans $\deg_{\mathbf{Q}} K = 1$, *then there exists a constant* $c_{19} > 0$ *such that for some* $y_0 \in Y(c_{19}S)$ *and some* t_0, $0 \leqslant t_0 \leqslant dT$,

$$F^{(t_0)}(y_0) \neq 0.$$

Proof. Define $X \subseteq G$ by $X = \phi(Y(c_{19}S))$. If the corollary is false, Théorème 2.1 of [5] implies that there exists a connected algebraic subgroup G^* of G such that

$$T \operatorname{card}\left(\frac{X + G^*}{G^*}\right) \leqslant \frac{H(G; (d_0 + \delta_a) L_0, d_1 L_1, 2d_g L_2, \delta_m l_1, 2\delta_g l_2)}{H(G^*; (d_0 + \delta_a) L_0, d_1 L_1, d_g L_2, \delta_m l_1, \delta_g l_2)}. \tag{14}$$

If G^* is trivial, then $H(G^*; (d_0 + \delta_a) L_0, d_1 L_1, d_g L_2, \delta_m l_1, \delta_g l_2) = 1$; $\operatorname{card}((X + G^*)/G^*) = \operatorname{card}(X) = (c_{19}S)^l$; and from (13) and (14) we obtain

$$T(c_{19}S)^l \leqslant C_G L_0^{d_0 + \delta_a} L_1^{d_1} L_2^{d_g} l_1^{\delta_m} l_2^{\delta_g}.$$

This inequality is contrary to our choice of parameters satisfying (12), provided c_{19} is taken to be sufficiently large. Hence we must deal with the case when G^* is nontrivial.

In general we may write $G^* = \mathbf{G}_a^e \times H_m \times H_g$ where H_m (respectively, H_g) is a connected algebraic subgroup of $\mathbf{G}_m^{d_1} \times \mathbf{G}_m^{\delta_m}$ (respectively, $G_2' \times G_2''$) and $0 \leqslant e \leqslant d_0 + \delta_a$. We claim that

$$H(G^*; (d_0+\delta_a) L_0, d_1 L_1, d_g L_2, \delta_m l_1, \delta_g l_2) \geqslant L_0^e L_2^{\dim G^* - e}. \tag{15}$$

To see this note that for some $\mathbf{a} = (a_1, ..., a_5)$ with $a_1 + \cdots + a_5 = \dim \mathbf{G}^*$, $\deg_{\mathbf{a}} G^* \neq 0$. Hence for any G^* we easily obtain $L_2^{\dim G^*}$ on the right-hand side of (15). To obtain the stronger result when $d_0 + \delta_a = e = 1$ we note that if $a_1 = 0$, then the corresponding $L_1 \subseteq P_{d_0+\delta_a}$ having codimension 0 yields either $G^* \cap L_1 \times \cdots \times L_5 = \varnothing$ or

$$\dim(G^* \cap L_1 \times \cdots \times L_5) \geqslant \dim(\mathbf{G}_a \cap L_1) = 1.$$

Hence $\deg_{\mathbf{a}} G^* \neq 0$ for some $\mathbf{a}$, with $a_1 = 1$; therefore (15) is established.

Let $r = \operatorname{cod}_G G^*$ and put $\hat{G} = G/G^*$; $\dim \hat{G} = r$. Moreover define $\hat{\phi}: \mathbf{C} \to \hat{G}(\mathbf{C})$ by taking ϕ composed with the canonical projection mapping. We then have $\hat{\phi}(Y) \subseteq \hat{G}(K)$ and $\hat{\phi}'(0) \in \mathscr{T}_{\hat{G}}(K)$.

If $\operatorname{card}((X+G^*)/G^*) = \operatorname{card}(X)$, then from (14) and (15) we obtain

$$T(c_{19} S)^l \leqslant C_G L_0^{d_0+\delta_a} L_1^{d_1} L_2^{d_g} l_1^{\delta_m} l_2^{\delta_g}$$

contrary to our choice of parameters (provided c_{19} is sufficiently large).

We assume, therefore, that $X \cap G^*$ is nontrivial; and it follows that $r_1 = 0$. The argument of [11, p. 298] yields the estimates

$$\operatorname{card}\left(\frac{X+G^*}{G^*}\right) \geqslant \begin{cases} (c_{19} S)^{l-1} & \text{if } r > 1 \\ (c_{19} S)^{l-2} & \text{if } r = 1. \end{cases}$$

Suppose $r > 1$. From (14) and (15) we deduce that

$$T(c_{19} S)^{l-1} \leqslant C_G S^{(\delta-2) r + (d_1+\delta_m)}.$$

Whence from our choice of parameters we obtain $r(\delta - 2) + (d_1 + \delta_m) \geqslant \delta + l - 1$; and since $l \geqslant \delta$, we deduce that

$$2l(r-1) \geqslant l + 2r + [l + 2r - 2(d_1 + \delta_m + 1)].$$

Recalling that we have assumed $d_1 + \delta_m \leqslant 2$, if we use $r \geqslant 2 \geqslant d_1 + \delta_m$, then $r - (d_1 + \delta_m + 1) \geqslant -1$, and from $l \geqslant 2$ it follows that

$$2l(r-1) \geqslant l + 2r. \tag{16}$$

We see by (11) and Theorem 2 of [11] (inequality (4) above) that trans $\deg_{\mathbf{Q}} K \geqslant 2$, contrary to our hypothesis.

If $r=1$, then (14) and (15) yield

$$T(c_{19}S)^{l-2} \leqslant C_G S^{\delta-2},$$

contrary to our choice of T above.

From a standard application of Schwarz's lemma on circles of radii $c_{20}S$ and $c_{20}S^{1+\varepsilon}$ for a small, positive ε, it follows that

$$\log |F^{(t_0)}(y_0)| < -c_{21} TS^l \log S.$$

Moreover, it then follows that there exists a nonzero polynomial $Q_S(x) \in \mathbb{Z}[x]$ with

$$\deg Q_S \leqslant c_{22}\delta(S), \qquad \log htQ_S \leqslant c_{23}\gamma(S)$$

such that

$$\log |Q_S(\theta)| < -c_{24} TS^l \log(S).$$

Since $c_{24} TS^l \log S > 6c_{22}\delta(S)(c_{22}\delta(S) + c_{23}\gamma(S))$ provided C_1 is sufficiently large, a routine application of Gelfond's Criterion for a number to be algebraic implies that θ is algebraic, contrary to the hypotheses above. This establishes Theorem 1.

Remark. By Théorème 1 of [14] the hypotheses of Theorem 1 imply that

$$l(\delta_a + \delta_m + \delta_g - 1) \leqslant \delta_m + 2\delta_g.$$

This restricts somewhat the applicability of this result.

4. Proof of Theorem 2

For the proof of Theorem 2 we define anew several quantities. Let

$$\delta = \frac{l + 2\delta_g + (d_1 + \delta_m)}{d-1} \qquad \text{and} \qquad \rho = \frac{d_1 + 1}{2(d-1)}.$$

Then for each real number $S > 0$ define L_0, L_1, T, l_1, l_2, and $S_1 = S \log^{1/2} S$,

$$L_0 = [\![S^\delta \log^\rho S]\!], \qquad L_1 = [\![S^{\delta-1} \log^{\rho-1/2} S]\!]$$

$$l_1 = [\![S^{\delta-1} \log^\rho S]\!], \qquad l_2 = [\![S^{\delta-2} \log^\rho S]\!], \qquad T = [\![\kappa_2 S^\delta \log^\rho S]\!]$$

with κ_2 chosen in the proof of Proposition 4.1 similar to how κ_1 was chosen in the proof of Proposition 3.2 above. Further let

$$\delta(S) = L_0 + L_1 S_1 + l_1 S + l_2 S^2 + T$$
$$\gamma(S) = L_0 \log S_1 + L_1 S_1 + l_1 S_1 + l_2 S_1^2 + T \log T.$$

To state the version of Proposition 3.2 which suffices, let

$$Y(\mathbf{S}) = y_1 \mathbb{Z}(S_1) + y_2 \mathbb{Z}(S) + \cdots + y_l \mathbb{Z}(S).$$

PROPOSITION 4.1. *If the hypotheses of Theorem* 2 *hold and* trans $\deg_{\mathbf{Q}} K = 1$, *then there exists a constant* $C_2 > 0$ *such that for all* $S > C_2$ *the following holds. There exists a multihomogeneous polynomial* $P(\mathbf{Z}_1, \mathbf{Y}_1, \mathbf{Z}_2, \mathbf{Y}_2, \mathbf{X}_2)$ *of multidegree* $(d_0 L_0, d_1 L_1, \delta_a L_0, \delta_m l_1, \delta_g l_2)$, *with coefficients in* $\mathcal{O}_K$ *of degree at most* $c_{25}\delta(S)$ *and logarithmic heights at most* $c_{26}\gamma(S)$, *such that*

$$F(z) = P \circ \phi(z)$$

satisfies

$$F^{(t)}(y) = 0, \qquad 0 \leqslant t < T,\ y \in Y(\mathbf{S}).$$

Moreover P does not vanish on all of G.

Proof. The proof is virtually identical to the proof of Proposition 3.2, hence we omit it.

Arguing as in the proof of Corollary 3.3 one then deduces that there exists a constant $c_{27} > 0$ such that for some $0 \leqslant t_0 \leqslant dT$ and some $y_0 \in Y(c_{27}\mathbf{S})$,

$$F^{(t_0)}(y_0) \neq 0.$$

Moreover, it follows from Schwarz's lemma that

$$\log |F^{(t_0)}(y_0)| < -c_{28} T S_1 S^{l-1} \log S.$$

This leads to the existence of a nonzero polynomial $Q_S(x) \in \mathbb{Z}[x]$ with

$$\deg Q_S \leqslant c_{29}\delta(S) \qquad \text{and} \qquad \log ht Q_S \leqslant c_{30}\gamma(S)$$

such that

$$\log |Q_S(\theta)| < -c_{31} T S_1 S^{l-1} \log S.$$

Note that by the choice of ρ,

$$\rho+\tfrac{3}{2}>2\rho+1,$$

hence Gelfond's Criterion implies that θ is algebraic, contrary to our hypotheses.

5. Proofs of Theorem 3 and Theorem 4

Although it is not entirely natural to do so, we prove Theorem 3 and Theorem 4 together. Under the hypotheses of each of these theorems it follows that trans $\deg_{\mathbf{Q}} K\geqslant 1$. Hence if we assume that these theorems are false, then in each case $K=\mathbf{Q}(\theta,\eta)$ where θ is transcendental and η is integral over $\mathbb{Z}[\theta]$.

We reintroduce the notion of Y-admissibility as defined in [11]. A pair (H,ϕ_H) which consists of a commutative algebraic group H and a one-parameter subgroup ϕ_H with $\phi_H(\mathbf{C})$ Zariski dense in H is *Y-admissible* with respect to a proposition if H, ϕ_H, and Y satisfy the hypotheses of the proposition. To prove Theorems 3 and 4 we assume that G is a commutative algebraic group of least dimension for which (G,ϕ) is Y-admissible with respect to the theorem under discussion.

For $S>0$ we redefine the real numbers L_0, L_1, D, T, and S_1 as follows. For the proof of Theorem 3 put

$$L_0=[\![S^3\log^{-1/4}S]\!],\qquad L_1=[\![S^2\log^{3/4}S]\!]$$
$$D=[\![S\log^{3/4}S]\!],\qquad T=[\![\kappa_3S^3\log^{-1/4}S]\!]$$

and $S_1=S^2\log^{3/2}S$ (as before $0<\kappa_3<1$ is chosen below).

For the proof of Theorem 4 put

$$L_0=[\![S^{3/2}]\!],\qquad L_1=[\![S^{3/4}]\!]$$
$$D=[\![S^{5/4}]\!],\qquad T=[\![\kappa_4S^{3/2}]\!]$$

and $S_1=S$ (with $0<\kappa_4<1$ chosen below).

We recall the definition of $Y(\mathbf{S})$, when $l=2$,

$$Y(\mathbf{S})=y_1\mathbb{Z}(S_1)+y_2\mathbb{Z}(S),$$

and that $\mathscr{G}$ denotes the multihomogeneous ideal which defines G in multiprojective space.

PROPOSITION 5.1. *Let G, ϕ, K, y_1, and y_2 be as in the statement of Theorem* 3 *(respectively, Theorem* 4*) and assume that* trans $\deg_{\mathbf{Q}} K=1$. *Then there exists a trihomogeneous polynomial*

$$P(\mathbf{Z}, \mathbf{Y}, \mathbf{X}) \in \mathcal{O}_K[\mathbf{Z}, \mathbf{Y}, \mathbf{X}] \backslash \mathcal{G}$$

of tridegree (L_0, L_1, D) and with coefficients in $\mathcal{O}_K$ of degree and logarithmic height at most $c_{32} S^3 \log^{3/4} S$ (respectively, $c_{33} S^{7/4}$) such that

$$F(z)=P \circ \phi(z)$$

satisfies

$$F^{(t)}(y)=0, \qquad 0 \leqslant t<T,\ y \in Y(\mathbf{S}).$$

Proof. Apply the same argument as in the proof of Proposition 3.2, taking into account the estimate for torsion points given by Lemma 3.1.

COROLLARY 5.2. *Let $F(z)$ be as above with Theorem* 3 *(respectively, Theorem* 4*) not holding. Then there exists a constant c_{34} such that for some $y_0 \in Y(c_{34} S)$ and some $0 \leqslant t_0 \leqslant dT$*

$$F^{(t_0)}(y_0) \neq 0.$$

Proof. View G as embedded in the multiprojective space $\mathbf{P}_{d_0} \times \mathbf{P}_{d_1} \times \mathbf{P}_N$ where $G_2 \hookrightarrow \mathbf{P}_N$ is the embedding given by Serre [9]. Let X denote the image of $Y(c_{34}\mathbf{S})$ under $\phi(z)$ composed with this embedding.

We first establish the corollary in the case of Theorem 3. If the corollary is false, then Théorème 2.1 of [5] implies that there exists a connected algebraic subgroup G^* of G with

$$G/G^*=\mathbf{G}_a^{r_0} \times \mathbf{G}_m^{r_1} \times H, \tag{17}$$

where $\dim H=r_2$, such that

$$T \operatorname{card}\left(\frac{X+G^*}{G^*}\right) \leqslant C_G L_0^{r_0} L_1^{r_1} D^{r_2} \tag{18}$$

(here C_G again depends on G and its embedding into multiprojective space). Let $\hat{\phi}: \mathbf{C} \rightarrow G/G^*(\mathbf{C})$ denote ϕ composed with the canonical projection mapping from G to G/G^*.

First suppose that $\operatorname{card}((X+G^*)/G^*)<\operatorname{card}(X)$. Therefore $\hat{\phi}$ is periodic and $r_0=0$. If $\operatorname{card}((X+G^*)/G^*)<S$, then $\hat{\phi}$ is doubly periodic, thus G/G^* is an elliptic curve so $r_0=0$, $r_1=0$, and $r_2=1$. The inequality (18) then becomes

$$T \leqslant c_{35} D$$

contrary to our choice of parameters, provided S is sufficiently large. Hence

$$\operatorname{card}\left(\frac{X+G^*}{G^*}\right) \geqslant S$$

and (18) together with an easy calculation considering the cases $r_1=0$, $r_1=1$, or $r_1 \geqslant 2$, yields $3r_1+2r_2 \geqslant 6$. Hence $\hat{\phi}: \mathbf{C} \to G/G^*(\mathbf{C})$ satisfies the hypothesis of Theorem 4 of [11] (inequality (4) above); and, therefore trans $\deg_{\mathbf{Q}} K \geqslant 2$ contrary to our assumption.

If $\operatorname{card}((X+G^*)/G^*) = \operatorname{card}(X)$ and $r_0=0$, then (18) cannot hold by our choice of parameters. Therefore $r_0=1$ and (18) yields $r_1+r_2 \geqslant 2$. This means that $(G/G^*, \hat{\phi})$ is Y-admissible with respect to Theorem 3; and, since G was chosen of minimal dimension such that Theorem 3 is false we know that if $\dim(G/G^*) < \dim G$, then we may conclude that trans $\deg_{\mathbf{Q}} K \geqslant 2$. Again this is contrary to our hypotheses. Hence G^* must be trivial, but then (18) cannot hold by our choice of parameters (provided c_{34} is sufficiently large). Therefore the corollary is established in the case of Theorem 3.

In the situation of Theorem 4 we again note that if the corollary is false, then there exists a connected algebraic subgroup G^* of G such that (17) and (18) hold. If

$$\operatorname{card}\left(\frac{X+G^*}{G^*}\right) < \operatorname{card}(X),$$

then $\hat{\phi}: \mathbf{C} \to G/G^*(\mathbf{C})$ is periodic, hence $r_0=0$. If, moreover, $\operatorname{card}((X+G^*)/G^*) < S$, then $\hat{\phi}$ is doubly periodic, G/G^* is an elliptic curve (so $r_1=0$, $r_2=1$), and by our choice of parameters (18) cannot hold. Therefore $\operatorname{card}((X+G^*)/G^*) \geqslant S$, so that (18) implies

$$\frac{3}{4} r_1 + \frac{5}{4} r_2 \geqslant \frac{5}{2}.$$

This last inequality cannot hold since $r_1 \leqslant 1$ and $r_2 \leqslant 1$.

Therefore, $\operatorname{card}((X+G^*)/G^*) = \operatorname{card}(X)$ and (18) yields

$$\frac{3}{4} r_1 + \frac{5}{4} r_2 \geqslant 2,$$

i.e., $r_1 = r_2 = 1$. Thus G^* is trivial and (18) cannot hold provided c_{34} is taken to be sufficiently large. This establishes the corollary in both cases.

From an application of Schwarz's lemma applied on circles of radii $c_{36} S_1$ and $c_{36} S_1^{1+\varepsilon}$ for a small positive ε, we deduce that

$$\log |F^{(t_0)}(y_0)| < -c_{37} T S S_1 \log S.$$

This leads to a nonzero polynomial $Q_S(x) \in \mathbb{Z}[x]$ with

$$\deg Q_S + \log htQ_S \leqslant c_{38} S^3 \log^{3/4} S \qquad (\text{resp. } \leqslant c_{39} S^{7/4})$$

such that $Q_S(\theta) \neq 0$ and

$$\log |Q_S(\theta)| < -c_{40} TSS_1 \log S.$$

By a now routine application of Gelfond's criterion this implies that θ is algebraic, contrary to our hypotheses.

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